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On Clifford algebras and unitary Lie superalgebras

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Abstract. Constructions of semidirect sums of $p[2p]$ Clifford algebras $Cl_{2n}[Cl_n]$ have recently appeared in supersymmetric developments. Moreover they are also related to specific properties on realizations of Green–Cusson's ansätze when parabosons of order p are concerned. We show that such semidirect sums are isomorphic to unitary Lie superalgebras. We also give a corresponding realization in terms of the basis elements of $\mathrm{Cl}_{2n+2q-2}[\mathrm{Cl}_{2m+2q-2}$, $n=2m$ or $n=2m-1]$. Each case is discussed according to the parity of p, i.e. $p=2q$ or $p=2q-1$.

1. Introduction

Supersymmetric formulations of quantum mechanics [1] have recently enhanced the role played by the so-called fermionic variables. These quantities are in fact issued from Clifford algebras [2] or from semidirect sums (i.e. the generators do not anticommute and give rise to non-trivial antisymmetric tensors) of two of such structures [2, 3] according to the different (already known) supersymmetrization procedures. In these simplest cases, it is easy to put in evidence the Clifford algebra Cl_{2n} when the standard procedure is used to describe n fermions [1, 3] while the semi-direct sum $\mathrm{Cl}_n \,\square\, \mathrm{Cl}_n$ naturally appears when the spin-orbit coupling procedure is under study [3]. The above two structures are not equivalent and it has recently been shown [4] that the last one is in fact intimately related to Lie unitary superalgebras [5].

The extension of statistics to parastatistics [6] and especially the use of Cusson's realization of the Green ansätze [7] when parabosons of order p are concerned have revealed new properties in supersymmetric quantum mechanics. Indeed let us consider the new supercharge [8]

$$Q=\sum_{j=1}^{n}\sum_{\mu=1}^{p} a_{\mu+(j-1)p}\xi_\mu \otimes \xi_{+,j} \tag{1.1}$$

where $a_{\mu+(j-1)p}$ is the usual (bosonic) annihilation operator, ξ_μ belongs to a Clifford algebra Cl_p and p is the order of paraquantization [6, 7]. The operators $A_j=\sum_{\mu=1}^{p} a_{\mu+(j-1)p}\xi_\mu$ $(j=1,\ldots,n)$ actually constitute a set of parabosonic annihilation operators [8].

It is well known [3, 4] that the odd quantities $\xi_{\pm,j}$ $(j=1,\ldots,n)$ have in general to satisfy

$$\{\xi_{+,j},\xi_{+,k}\}=\{\xi_{-,j},\xi_{-,k}\}=0 \tag{1.2a}$$

$$\{\xi_{+,j},\xi_{-,k}\}=\delta_{jk}-\mathrm{i}\Xi_{jk} \tag{1.2b}$$

$$(\xi_{+,j})^\dagger=\xi_{-,j} \qquad \Xi_{jk}=-\Xi_{kj} \qquad \Xi^\dagger=\Xi. \tag{1.2c}$$

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By defining the new *Hermitian* operators

$$\beta_j = \xi_{+,j} + \xi_{-,j} \qquad \beta_{j+n} = \mathrm{i}(\xi_{-,j} - \xi_{+,j}) \tag{1.3}$$

the relations (1.2a) and (1.2b) can also be written

$$\{\beta_j, \beta_k\} = 2\delta_{jk} \qquad \{\beta_{j+n}, \beta_{k+n}\} = 2\delta_{jk} \tag{1.4a}$$

$$\{\beta_j, \beta_{k+n}\} = 2\Xi_{jk}. \tag{1.4b}$$

As a consequence, these operators generate the already mentioned structures Cl_{2n} *or* $\mathrm{Cl}_n \square \mathrm{Cl}_n$ if the antisymmetric matrices Ξ_{jk} are identically zero *or* not, respectively. We just recall here that, in the second case (corresponding to the spin–orbit coupling supersymmetrization procedure as already mentioned), it is possible to reduce [9] the dimension of the matrix realization by imposing linear dependences between the fermionic quantities. They will lead to

$$\Xi_{jk} = -\delta_{k,j+1} \tag{1.5}$$

if we choose

$$\beta_{2j} = \beta_{n+2j-1} \qquad \beta_{2j-1} = -\beta_{n+2j} \qquad (j = 1, \ldots, m;\ n = 2m \text{ or } n = 2m-1). \tag{1.6}$$

This clearly announces that we have to take care of the parameter m in this case while it is not necessary in the standard developments.

Moreover, if, by coming back on the parastatistical point of view subtended here, *all* the $2np$ fermionic quantities $\xi_\mu \otimes \xi_{\pm,j}$ or equivalently the sets $\{\xi_\mu \otimes \beta_j\}$ *and* $\{\xi_\mu \otimes \beta_{j+n}\}$ have to be considered, let us recall that we have pointed out [8] that the above structures are always extended to the semi-direct sums of p terms

$$\mathrm{Cl}_{2n} \square \mathrm{Cl}_{2n} \square \ldots \square \mathrm{Cl}_{2n} \tag{1.7a}$$

or

$$[\mathrm{Cl}_n \square \mathrm{Cl}_n] \square [\mathrm{Cl}_n \blacksquare \mathrm{Cl}_n] \square \ldots \square [\mathrm{Cl}_n \square \mathrm{Cl}_n] \tag{1.7b}$$

respectively, according to the cancellation or not of the Ξ_{jk} s. The particular case $p = 1$ which precisely corresponds to the non-para context leads to (1.4) only. The operators $\xi_\mu \otimes \beta_j$, $\xi_\mu \otimes \beta_{j+n}$ where j runs from 1 to n and μ is fixed, generate fixed Cl_{2n} in (1.7a) and a fixed $\mathrm{Cl}_n \square \mathrm{Cl}_n$ in (1.7b). They give rise to a *new* supersymmetric Hamiltonian [8] defined as usual by the anticommutator of $Q(\equiv(1.1))$ and its Hermitian conjugate.

We thus obtain

$$\begin{aligned} H_{SS} = {} & \tfrac{1}{2} \sum_{j=1}^{n} \sum_{\mu=1}^{p} \{a_{\mu+(j-1)p}, a^\dagger_{\mu+(j-1)p}\} + \frac{p}{2} \sum_{j=1}^{n} I_{2^q} \otimes [\xi_{+,j}, \xi_{-,j}] \\ & + \tfrac{1}{2} \sum_{j,k=1}^{n} \sum_{\mu \neq \nu}^{p} (a_{\mu+(j-1)p} a^\dagger_{\nu+(k-1)p} - a_{\nu+(j-1)p} a^\dagger_{\mu+(k-1)p}) \xi_\mu \xi_\nu \\ & \otimes [\xi_{+,j}, \xi_{-,k}] - \tfrac{1}{2} \sum_{j \neq k}^{n} \sum_{\mu=1}^{p} \{a_{\mu+(j-1)p}, a^\dagger_{\mu+(k-1)p}\} I_{2^q} \otimes \Xi_{jk} \end{aligned} \tag{1.8}$$

if $p = 2q$ or $p = 2q - 1$. It superposes np bosons and np fermions and *all* the already studied Hamiltonians [3, 10] are recovered when $p = 1$. Consequently, we will mainly consider in the following Section the new case $p \geq 2$ related to parastatistical considerations, the effective purpose of this paper.

2. The case $p \geqslant 2$

The fermionic variables associated with the general case $p \geqslant 2$ are not easily realized but they are simply related to unitary Lie superalgebras as we want now to show through the following two propositions (in correspondence with the semidirect sums (1.7a) and (1.7b) respectively). Let us consider successively the standard *and* spin-orbit coupling contexts.

2.1. The standard context

Let us start with $(2n+2q-2)$ basis elements α_k generating a Clifford algebra $\mathrm{Cl}_{2n+2q-2}$, i.e. satisfying

$$\{\alpha_j, \alpha_k\} = 2\delta_{jk} \qquad (j, k = 1, \ldots, 2n+2q-2). \tag{2.1}$$

Lemma 1. If $p = 2q-1$ (>1), the fermionic operators $\xi_\mu \otimes \beta_j$, $\xi_\mu \otimes \beta_{j+n}$ can be realized as follows when $\{\beta_j, \beta_{k+n}\} = 0$:

$$\xi_\mu \otimes \beta_1 = \alpha_\mu \tag{2.2a}$$

$$\xi_\mu \otimes \beta_j = \mathrm{i}^r(-1)^{\mu+1}\alpha_1 \ldots [\alpha_\mu] \ldots \alpha_{2q-1}\alpha_{j+2q-2} \qquad (j = 2, \ldots, n) \tag{2.2b}$$

$$\xi_\mu \otimes \beta_{j+n} = \mathrm{i}^r(-1)^{\mu+1}\alpha_1 \ldots [\alpha_\mu] \ldots \alpha_{2q-1}\alpha_{j+n+2q-2} \qquad (j = 1, \ldots, n) \tag{2.2c}$$

where $\mu = 1, \ldots, p$, $r = 1$ (2) if q is even (odd). If $p = 2q$, the supplementary following generators have to be added to the previous ones:

$$\xi_{2q} \otimes \beta_1 = \mathrm{i}^s\alpha_{2q} \ldots \alpha_{2n+2q-2} \tag{2.3a}$$

$$\xi_{2q} \otimes \beta_j = \mathrm{i}^{r+s}(-1)^{j+1}\alpha_1 \ldots [\alpha_{j+2q-2}] \ldots \alpha_{2n+2q-2} \qquad (j = 2, \ldots, n) \tag{2.3b}$$

$$\xi_{2q} \otimes \beta_{j+n} = \mathrm{i}^{r+s}(-1)^{j+1+n}\alpha_1 \ldots [\alpha_{j+n+2q-2}] \ldots \alpha_{2n+2q-2} \qquad (j = 1, \ldots, n) \tag{2.3c}$$

where $s = 1$ (2) if n is even (odd). Notice that the notation $[k]$ inside a product means that the factor k is missing.

Proof. The following anticommutation relations (2.4) are easily determined and are sufficient to prove our assertion if $p = 2q-1$:

$$\{\xi_\mu \otimes \beta_1, \xi_\nu \otimes \beta_1\} = \{\xi_\mu \otimes \beta_j, \xi_\nu \otimes \beta_j\} = \{\xi_\mu \otimes \beta_{j+n}, \xi_\nu \otimes \beta_{j+n}\} = 2\delta_{\mu\nu} \tag{2.4a}$$

$$\begin{aligned}\{\xi_\mu \otimes \beta_1, \xi_\nu \otimes \beta_j\} &= 2\mathrm{i}^r(-1)^{\mu+\nu}\alpha_1 \ldots [\alpha_\mu] \ldots [\alpha_\nu] \ldots \alpha_{2q-1}\alpha_{j+2q-2} \\ &= -\{\xi_\nu \otimes \beta_1, \xi_\mu \otimes \beta_j\}\end{aligned} \tag{2.4b}$$

$$\begin{aligned}\{\xi_\mu \otimes \beta_1, \xi_\nu \otimes \beta_{j+n}\} &= 2\mathrm{i}^r(-1)^{\mu+\nu}\alpha_1 \ldots [\alpha_\mu] \ldots [\alpha_\nu] \ldots \alpha_{2q-1}\alpha_{j+n+2q-2} \\ &= -\{\xi_\nu \otimes \beta_1, \xi_\mu \otimes \beta_{j+n}\}\end{aligned} \tag{2.4c}$$

$$\{\xi_\mu \otimes \beta_j, \xi_\nu \otimes \beta_k\} = -2\alpha_\mu\alpha_\nu\alpha_{j+2q-2}\alpha_{k+2q-2} = -\{\xi_\nu \otimes \beta_j, \xi_\mu \otimes \beta_k\} \qquad (j \neq k) \tag{2.4d}$$

$$\{\xi_\mu \otimes \beta_j, \xi_\nu \otimes \beta_{k+n}\} = -2\alpha_\mu\alpha_\nu\alpha_{j+2q-2}\alpha_{k+n+2q-2} = -\{\xi_\nu \otimes \beta_j, \xi_\mu \otimes \beta_{k+n}\} \tag{2.4e}$$

$$\begin{aligned}\{\xi_\mu \otimes \beta_{j+n}, \xi_\nu \otimes \beta_{k+n}\} &= -2\alpha_\mu\alpha_\nu\alpha_{j+n+2q-2}\alpha_{k+n+2q-2} \\ &= -\{\xi_\nu \otimes \beta_{j+n}, \xi_\mu \otimes \beta_{k+n}\} \qquad (j \neq k).\end{aligned} \tag{2.4f}$$

The case of even p is settled by computing the non-vanishing relations (2.5) and (2.6):

$$\{\xi_{2q}\otimes\beta_1, \xi_{2q}\otimes\beta_1\}=2 \tag{2.5a}$$

$$\{\xi_{2q}\otimes\beta_j, \xi_{2q}\otimes\beta_k\}=\{\xi_{2q}\otimes\beta_{j+n}, \xi_{2q}\otimes\beta_{k+n}\}=2\delta_{jk} \tag{2.5b}$$

$$\begin{aligned}\{\xi_\mu\otimes\beta_1, \xi_{2q}\otimes\beta_j\}&=2\mathrm{i}^{r+s}(-1)^{j+\mu}\alpha_1\ldots[\alpha_\mu]\ldots[\alpha_{j+2q-2}]\ldots\alpha_{2n+2q-2}\\ &=-\{\xi_{2q}\otimes\beta_1, \xi_\mu\otimes\beta_j\}\end{aligned} \tag{2.6a}$$

$$\begin{aligned}\{\xi_\mu\otimes\beta_1, \xi_{2q}\otimes\beta_{j+n}\}&=2\mathrm{i}^{r+s}(-1)^{j+\mu+n}\alpha_1\ldots[\alpha_\mu]\ldots[\alpha_{j+n+2q-2}]\ldots\alpha_{2n+2q-2}\\ &=-\{\xi_{2q}\otimes\beta_1, \xi_\mu\otimes\beta_{j+n}\}\end{aligned} \tag{2.6b}$$

$$\begin{aligned}\{\xi_\mu\otimes\beta_j, \xi_{2q}\otimes\beta_k\}&=2\mathrm{i}^{s}(-1)^{j+k}\alpha_\mu\alpha_{2q}\ldots[\alpha_{j+2q-2}]\ldots[\alpha_{k+2q-2}]\ldots\alpha_{2n+2q-2}\\ &=-\{\xi_{2q}\otimes\beta_j, \xi_\mu\otimes\beta_k\}\qquad (j\neq k)\end{aligned} \tag{2.6c}$$

$$\begin{aligned}\{\xi_\mu\otimes\beta_j, \xi_{2q}\otimes\beta_{k+n}\}&=2\mathrm{i}^{s}(-1)^{j+k+n}\alpha_\mu\alpha_{2q}\ldots[\alpha_{j+2q-2}]\ldots[\alpha_{k+n+2q-2}]\ldots\alpha_{2n+2q-2}\\ &=-\{\xi_{2q}\otimes\beta_j, \xi_\mu\otimes\beta_{k+n}\}\end{aligned} \tag{2.6d}$$

$$\begin{aligned}\{\xi_\mu\otimes\beta_{j+n}, \xi_{2q}\otimes\beta_{k+n}\}&=2\mathrm{i}^{s}(-1)^{j+k}\alpha_\mu\alpha_{2q}\ldots[\alpha_{j+n+2q-2}]\ldots[\alpha_{k+n+2q-2}]\ldots\alpha_{2n+2q-2}\\ &=-\{\xi_{2q}\otimes\beta_{j+n}, \xi_\mu\otimes\beta_{k+n}\}\qquad (j\neq k).\end{aligned} \tag{2.6e}$$

Equations (2.5) mean that we obtain *another* Cl_{2n} and this Clifford algebra is in semidirect sum with the other ones in a right way (see (2.6)).

Proposition 1. The semidirect sum of p ($=2q-1$ or $2q$) terms $\mathrm{Cl}_{2n}\,\square\,\mathrm{Cl}_{2n}\,\square\ldots\square\,\mathrm{Cl}_{2n}$ is isomorphic to the unitary superalgebra $\mathrm{su}(2^{n+q-2}|2^{n+q-2})$.

Proof. Let us start with the realization given in lemma 1 when $p=2q-1$. It gives rise to even operators quoted in (2.4b-f). Their commutation relations put in evidence new even generators realized as $\alpha_\mu\alpha_\nu$, $\alpha_{j+2q-2}\alpha_{k+2q-2}$, $\alpha_{j+2q-2}\alpha_{k+n+2q-2}$, $\alpha_{j+n+2q-2}\alpha_{k+n+2q-2}$ and, in general,

$$\alpha_1\ldots[\alpha_\mu]\ldots[\alpha_\nu]\ldots[\alpha_\rho]\ldots[\alpha_\tau]\ldots\alpha_{2q-1}\alpha_{k+2q-2}$$

$\alpha_1\ldots[\alpha_\mu]\ldots[\alpha_\nu]\ldots[\alpha_\rho]\ldots[\alpha_\tau]\ldots\alpha_{2q-1}\alpha_{k+n+2q-2}$ where we have omitted $2t$ ($t=1,\ldots,q-1$) terms. In particular, we obtain by this way $\alpha_\mu\alpha_{k+2q-2}$ and $\alpha_\mu\alpha_{k+n+2q-2}$. We have thus recovered all the $(n+q-1)(2n+2q-3)$ possible products of two α's. By commutation relations with the previous α_μ ($\mu=1,\ldots,2q-1$), we get $\alpha_{2q},\ldots,\alpha_{2n+2q-2}$. Going on with this procedure, we can obtain all the $2^{2n+2q-2}$ elements belonging to the vectorial space subtended by $\mathrm{Cl}_{2n+2q-2}$ except the so-called canonical element [4, 11]

$$\Lambda=\alpha_1\ldots\alpha_{2n+2q-2}. \tag{2.7}$$

We thus recover the corresponding conditions of [4] and we can deduce that these operators generate the superalgebra $\mathrm{su}(2^{n+q-2}|2^{n+q-2})$. In particular, the generators corresponding to the $p=2q$-case are included into this superstructure. This concludes the proof.

2.2. *The spin-orbit coupling context*

The second structure (1.7b) can be studied in a perfectly similar way and we just propose to quote the results. By recalling that we have to take care of m instead of the previous n (see (1.6)), let us start here with $(2m+2q-2)$ basis elements α_k of a Clifford algebra $\mathrm{Cl}_{2m+2q-2}$.

Lemma 2. If $p=2q-1$ (>1), the fermionic operators $\xi_\mu \otimes \beta_j$, $\xi_\mu \otimes \beta_{j+n}$ can be realized as follows when $\{\beta_j, \beta_{k+n}\}=2\Xi_{jk}$:

$$\xi_\mu \otimes \beta_1 = -\xi_\mu \otimes \beta_{n+2} = \alpha_\mu \tag{2.8a}$$

$$\xi_\mu \otimes \beta_{2j} = \xi_\mu \otimes \beta_{n+2j-1} = \mathrm{i}^r(-1)^{\mu+1}\alpha_1 \ldots [\alpha_\mu] \ldots \alpha_{2q-1}\alpha_{2j+2q-2} \qquad (j=1,\ldots,m) \tag{2.8b}$$

$$\xi_\mu \otimes \beta_{2j-1} = -\xi_\mu \otimes \beta_{n+2j} = \mathrm{i}^r(-1)^{\mu+1}\alpha_1 \ldots [\alpha_\mu] \ldots \alpha_{2q-1}\alpha_{2j+2q-3} \qquad (j=2,\ldots,m) \tag{2.8c}$$

where $\mu=1,\ldots,p$, $r=1$ (2) if q is even (odd). If $p=2q$, the generators (2.9) have to be added to the previous ones:

$$\xi_{2q} \otimes \beta_1 = -\xi_{2q} \otimes \beta_{n+2} = \mathrm{i}^s \alpha_{2q} \ldots \alpha_{2m+2q-2} \tag{2.9a}$$

$$\xi_{2q} \otimes \beta_{2j} = \xi_{2q} \otimes \beta_{n+2j-1} = -\mathrm{i}^{r+s}\alpha_1 \ldots [\alpha_{2j+2q-2}] \ldots \alpha_{2m+2q-2} \qquad (j=1,\ldots,m) \tag{2.9b}$$

$$\xi_{2q} \otimes \beta_{2j-1} = -\xi_{2q} \otimes \beta_{n+2j} = \mathrm{i}^{r+s}\alpha_1 \ldots [\alpha_{2j+2q-3}] \ldots \alpha_{2m+2q-2} \qquad (j=2,\ldots,m) \tag{2.9c}$$

where $s=1$ (2) if m is even (odd).

Proposition 2. The semidirect sum of p $(=2q-1$ or $2q)$ terms $[\mathrm{Cl}_n \,\square\, \mathrm{Cl}_n]$ $\square \ldots \square [\mathrm{Cl}_n \,\square\, \mathrm{Cl}_n]$ is isomorphic to the unitary superalgebra $\mathrm{su}(2^{m+q-2}|2^{m+q-2})$ if $n=2m$ or $n=2m-1$.

As a final comment, let us notice that the order of $\mathrm{su}(2^{n+q-2}|2^{n+q-2})$ $[\mathrm{su}(2^{m+q-2}|2^{m+q-2})]$ is $(2^{2n+2q-2}-1)$ $[(2^{2m+2q-2}-1)]$. It corresponds to the dimension of the vectorial space spanned by $\mathrm{Cl}_{2n+2q-2}[\mathrm{Cl}_{2m+2q-2}]$ where the canonical element $\Lambda \equiv (2.7)$ has been omitted as it cannot be generated through the structure relations between odd and even operators. However, let us mention that if we would have taken care of Λ, we would have recovered the superalgebra $\mathrm{u}(2^{n+q-2}|2^{n+q-2})[\mathrm{u}(2^{m+q-2}|2^{m+q-2})]$.

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